6

SEQUENCES AND SERIES

Succession of numbers of which one number is designated as the first, other as the second, another as the third and so on gives rise to what is called a sequence. Sequences have wide applications. In this lesson we shall discuss particular types of sequences called arithmetic sequence, geometric sequence and also find arithmetic mean (A.M), geometric mean (G.M) between two given numbers. We will also establish the relation between A.M and G.M.

Let us consider the following problems :

(a) A man places a pair of newly born rabbits into a warren and wants to know how many rabbits he would have over a certain period of time. A pair of rabbits will start producing offsprings two months after they were born and every following month one new pair of rabbits will appear. At the beginning the man will have in his warren only one pair of rabbits, during the second month he will have the same pair of rabbits, during the third month the number of pairs of rabbits in the warren will grow to two; during the fourth month there will be three pairs of rabbits in the warren. Thus, the number of pairs of rabbits in the consecutive months are :

1, 1, 2, 3, 5, 8, 13, ...

(b) The recurring decimal $0.\bar{3}$ can be written as a sum

$0.\bar{3}$ = 0.3 + 0.03 + 0.003 + 0.0003 ...

(c) A man earns Rs.10 on the first day, Rs. 30 on the second day, Rs. 50 on the third day and so on. The day to day earning of the man may be written as 10, 30, 50, 70, 90, $\cdots$

We may ask what his earnings will be on the 10^{th} day in a specific month.

Again let us consider the following sequences:

(1) 2, 4, 8, 16, $\cdots$ (2) $\frac{1}{9}, -\frac{1}{27}, \frac{1}{81}, -\frac{1}{243}, \ldots$

(3) 0.01, 0.0001, 0.000001, $\cdots$

In these three sequences, each term except the first, progressess in a definite order but different from the order of other three problems. In this lesson we will discuss those sequences whose term progressess in a definite order.

OBJECTIVES

After studying this lesson, you will be able to :

- describe the concept of a sequence (progression);
- define an A.P. and cite examples;
- find common difference and general term of a A.P;
- find the fourth quantity of an A.P. given any three of the quantities a, d, n and t_n;
- calculate the common difference or any other term of the A.P. given any two terms of the A.P;
- derive the formula for the sum of 'n' terms of an A.P;
- calculate the fourth quantity of an A.P. given three of S, n, a and d;
- insert A.M. between two numbers;
- solve problems of daily life using concept of an A.P;
- state that a geometric progression is a sequence increasing or decreasing by a definite multiple of a non-zero number other than one;
- identify G.P.'s from a given set of progessions;
- find the common ratio and general term of a G.P;
- calculate the fourth quantity of a G.P when any three of the quantities t_n, a, r and n are given;
- calculate the common ratio and any term when two of the terms of the G.P. are given;
- write progression when the general term is given;
- derive the formula for sum of n terms of a G.P;
- calculate the fourth quantity of a G.P. if any three of a, r, n and S are given;
- derive the formula for sum (S_∞) of infinite number of terms of a G.P. when $|r| < 1$;
- find the third quantity when any two of S_∞, a and r are given;
- convert recurring decimals to fractions using G.P;
- insert G.M. between two numbers; and
- establish relationship between A.M. and G.M.

EXPECTED BACKGROUND KNOWLEDGE

- Laws of indices
- Simultaneous equations with two unknowns.
- Quadratic Equations.

6.1 SEQUENCE

A sequence is a collection of numbers specified in a definite order by some assigned law, whereby a definite number a_n of the set can be associated with the corresponding positive integer n. The different notations used for a sequence are.

1. $a_1, a_2, a_3, \ldots, a_n, \ldots.$ 2. a_n , $n = 1, 2, 3, \ldots$ 3. $\{a_n\}$

Let us consider the following sequences :

1. 1, 2, 4, 8, 16, 32, ... 2. 1, 4, 9, 16, 25, ...

3. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots$ 4. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \cdots$

In the above examples, the expression for n^{th} term of the sequences are as given below :

(1) $a_n = 2^{n-1}$ (2) $a_n = n^2$ (3) $a_n = \frac{n}{n+1}$ (4) $a_n = \frac{1}{n}$

for all positive integer n.

Also for the first problem in the introduction, the terms can be obtained from the relation

$$a_1 = 1,\ a_2 = 1,\ a_n = a_{n-2} + a_{n-1},\ \text{n} \geq 3$$

A finite sequence has a finite number of terms. An infinite sequence contains an infinite number of terms.

6.2 ARITHMETIC PROGRESSION

Let us consider the following examples of sequence, of numbers :

(1) $2, 4, 6, 8, \cdots$ (2) $1, \frac{3}{2}, 2, \frac{5}{2}, \cdots$

(3) $10, 8, 6, 4, \cdots$ (4) $-\frac{1}{2}, -1, -\frac{3}{2}, -2, -\frac{5}{2}, \cdots$

Note that in the above four sequences of numbers, the first terms are respectively 2, 1, 10, and $-\frac{1}{2}$. The first term has an important role in this lesson. Also every following term of the sequence has certain relation with the first term. What is the relation of the terms with the first term in Example (1) ? First term = 2, Second term = 4 = 2 + 1 × 2

Third term = 6 = 2 + 2 × 2

Fourth term = 8 = 2 + 3 × 2 and so on.

The consecutive terms in the above sequence are obtained by adding 2 to its preceding term. i.e., the difference between any two consecutive terms is the same.

A finite sequence of numbers with this property is called an arithmetic progression.
A sequence of numbers with finite terms in which the difference between two consecutive terms is the same non-zero number is called the Arithmetic Progression or simply A. P.

The difference between two consecutive terms is called the common defference of the A. P. and is denoted by 'd'.

In general, an A. P. whose first term is a and common difference is d is written as $a, a + d, a + 2d, a + 3d, \cdots$

Also we use t_n to denote the nth term of the progression.

6.2.1 GENERAL TERM OF AN A. P.

Let us consider A. P. $a, a + d, a + 2d, a + 3d, \cdots$

Here, first term $(t_1) = a$

second term $(t_2) = a + d = a + (2 - 1)\, d,$

third term $(t_3) = a + 2d = a + (3 - 1)\, d$

By observing the above pattern, n[th] term can be written as: $t_n = a + (n - 1)\, d$

Hence, if the first term and the common difference of an A. P. are known then any term of A. P. can be determined by the above formula.

Note.:

(i) If the same non-zero number is added to each term of an A. P. the resulting sequence is again an A. P.

(ii) If each term of an A. P. is multiplied by the same non-zero number, the resulting sequence is again an A. P.

Example 6.1 Find the 10[th] term of the A. P.: 2, 4, 6, ...

Solution : Here the first term $(a) = 2$ and common difference $d = 4 - 2 = 2$

Using the formula $t_n = a + (n - 1)\, d$, we have

$$t_{10} = 2 + (10 - 1)\, 2 = 2 + 18 = 20$$

Hence, the 10th term of the given A. P. is 20.

Example 6.2 The 10[th] term of an A. P. is – 15 and 31[st] term is –57, find the 15[th] term.

Solution : Let a be the first term and d be the common difference of the A. P. Then from the formula: $t_n = a + (n - 1)\, d$, we have

$$t_{10} = a + (10 - 1)\, d = a + 9d \text{ and } t_{31} = a + (31 - 1)\, d = a + 30\, d$$

We have, $a + 9d = -15$...(1), $a + 30d = -57$ ...(2)

Solve equations (1) and (2) to get the values of a and d.

Subtracting (1) from (2), we have

$$21d = -57 + 15 = -42 \qquad \therefore\ d = \frac{-42}{21} = -2$$

Again from (1), $a = -15 - 9d = -15 - 9(-2) = -15 + 18 = 3$

Now $t_{15} = a + (15 - 1)d = 3 + 14(-2) = -25$

Example 6.3 Which term of the A. P.: 5, 11, 17, ... is 119 ?

Solution : Here $a = 5$, $d = 11 - 5 = 6$

$$t_n = 119$$

We know that $t_n = a + (n - 1)\,d$

$$\Rightarrow \quad 119 = 5 + (n-1) \times 6 \quad \Rightarrow \quad (n-1) = \frac{119 - 5}{6} = 19$$

$$\therefore \quad n = 20$$

Therefore, 119 is the 20th term of the given A. P.

Example 6.4 Is 600 a term of the A. P.: 2, 9, 16, ...?

Solution : Here, $a = 2$, and $d = 9 - 2 = 7$.

Let 600 be the n^{th} term of the A. P. We have $t_n = 2 + (n - 1)\,7$

According to the question,

$$2 + (n-1)\,7 = 600 \qquad \therefore\ (n-1)\,7 = 598$$

or $$n = \frac{598}{7} + 1 \qquad \therefore\ n = 86\frac{3}{7}$$

Since n is a fraction, it cannot be a term of the given A. P. Hence, 600 is not a term of the given A. P.

Example 6.5 If $a + b + c = 0$ and $\dfrac{a}{b+c}, \dfrac{b}{c+a}, \dfrac{c}{a+b}$ are. in A. P., then prove that

$\dfrac{1}{b+c}, \dfrac{1}{c+a}, \dfrac{1}{a+b}$ are also in A. P.

Solution. : Since $\dfrac{a}{b+c}, \dfrac{b}{c+a}, \dfrac{c}{a+b}$ are in A. P., therefore

$$\frac{b}{c+a} - \frac{a}{b+c} = \frac{c}{a+b} - \frac{b}{c+a}$$

or, $$\left(\frac{b}{c+a}+1\right) - \left(\frac{a}{b+c}+1\right) = \left(\frac{c}{a+b}+1\right) - \left(\frac{b}{c+a}+1\right)$$

or, $$\frac{a+b+c}{c+a} - \frac{a+b+c}{b+c} = \frac{a+b+c}{a+b} - \frac{a+b+c}{c+a}$$

or, $$\frac{1}{c+a} - \frac{1}{b+c} = \frac{1}{a+b} - \frac{1}{c+a} \quad \text{(Since } a+b+c \neq 0\text{)}$$

or, $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A. P.

CHECK YOUR PROGRESS 6.1

1. Find the n^{th} term of each of the following A. P's. :

   (a) $1, 3, 5, 7, \cdots$ (b) $3, 5, 7, 9, \cdots$

2. If $t_n = 2n + 1$, then find the A. P.

3. Which term of the A. P. $2\frac{1}{2}, 4, 5\frac{1}{2}, \ldots\ldots$ is 31? Find also the 10^{th} term?

4. Is – 292 a term of the A. P. 7, 4, 1, –2, ...?

5. The m^{th} term of an A. P. is n and the n^{th} term is m. Show that its $(m + n)^{th}$ term is zero.

6. Three numbers are in A. P. The difference between the first and the last is 8 and the product of these two is 20. Find the numbers.

7. The n^{th} term of a sequence is $na + b$. Prove that the sequence is an A. P. with common difference a.

6.3 TO FIND THE SUM OF FIRST *n* TERMS IN AN A. P.

Let a be the first term and d be the common difference of an A. P. Let l denote the last term, i.e., the n^{th} term of the A. P. Then, $l = t_n = a + (n-1)d$ ... (i)

Let S_n denote the sum of the first n terms of the A. P. Then

$$S_n = a + (a + d) + (a + 2d) + \ldots + (l - 2d) + (l - d) + l \qquad \text{... (ii)}$$

Reversing the order of terms in the R. H. S. of the above equation, we have

$$S_n = l + (l - d) + (l - 2d) + \ldots + (a + 2d) + (a + d) + a \qquad \text{... (iii)}$$

Adding (ii) and (iii) vertically, we get

$$2S_n = (a + l) + (a + l) + (a + l) + \ldots \text{ containing } n \text{ terms} = n\,(a + l)$$

i.e., $S_n = \dfrac{n}{2}(a + l)$

Also $S_n = \dfrac{n}{2}[2a + (n-1)\,d]$ [From (i)]

It is obvious that $t_n = S_n - S_{n-1}$

Example 6.6 Find the sum of $2 + 4 + 6 + \ldots n$ terms.

Solution.: Here $a = 2$, $d = 4 - 2 = 2$

Using the formula $S_n = \dfrac{n}{2}[2a + (n-1)\,d]$, we get

$$S_n = \frac{n}{2}[2 \times 2 + (n-1)\,2] = \frac{n}{2}[2 + 2n] = \frac{2n\,(n+1)}{2} = n\,(n + 1)$$

Example 6.7 The 35th term of an A. P. is 69. Find the sum of its 69 terms.

Solution. Let a be the first term and d be the common difference of the A. P.

We have $t_{35} = a + (35 - 1)\,d = a + 34\,d$.

$\therefore \quad a + 34\,d = 69 \qquad \ldots$ (i)

Now by the formula, $S_n = \dfrac{n}{2}[2a + (n-1)\,d]$

We have $S_{69} = \dfrac{69}{2}[2a + (69-1)\,d]$

$= 69\,(a + 34d)$ [using (i)]

$= 69 \times 69 = 4761$

Example 6.8 The first term of an A. P. is 10, the last term is 50. If the sum of all the terms is 480, find the common difference and the number of terms.

Solution : We have: $a = 10$, $l = t_n = 50$, $S_n = 480$.

By substituting the values of a, t_n and S_n in the formulae

$$S_n = \frac{n}{2}\left[2a+(n-1)d\right] \text{ and } t_n = a+(n-1)d, \text{ we get}$$

$$480 = \frac{n}{2}\left[20+(n-1)d\right] \qquad \cdots \text{(i)}$$

$$50 = 10 + (n-1)d \qquad \cdots \text{(ii)}$$

From (ii), $(n-1)d = 50 - 10 = 40$ $\qquad \cdots$ (iii)

From (i), we have $480 = \frac{n}{2}(20+40)$ $\qquad$ using (i)

or, $\quad 60n = 2 \times 480 \quad \therefore \quad n = \frac{2\times 480}{60} = 16$

From (iii),

$$\therefore \quad d = \frac{40}{15} = \frac{8}{3} \qquad (\text{as } n-1=16-1=15)$$

Example 6. 9 Let the n^{th} term and the sum of n terms of an A. P. be p and q respectively.

Prove that its first term is $\left(\frac{2q-pn}{n}\right)$.

Solution: In this case, $t_n = p$ and $S_n = q$

Let a be the first term of the A. P.

Now, $S_n = \frac{n}{2}(a+t_n)$ $\quad$ or, $\quad \frac{n}{2}(a+p)=q$

or, $\quad a+p = \frac{2q}{n}$ $\quad$ or, $\quad a = \frac{2q}{n} - p$ $\quad \therefore \quad a = \frac{2q-pn}{n}$

CHECK YOUR PROGRESS 6.2

1. Find the sum of the following A. P's.

   (a) 8, 11, 14, 17, $\cdots$ up to 15 terms $\qquad$ (b) 8, 3, –2, –7, –12, $\cdots$ up to n terms.

2. How many terms of the A. P.: 27, 23, 19, 15, ... have a sum 95?

3. A man takes an interest-free loan of Rs. 1740 from his friend agreeing to repay in monthly instalments. He gives Rs. 200 in the first month and diminishes his monthly instalments by Rs. 10 each month. How many months will it take to repay the loan?

4. How many terms of the progression 3, 6, 9, 12, $\cdots$

   must be taken at the least to have a sum not less than 2000?

5. In a children potato race, n potatoes are placed 1 metre apart in a straight line. A competitor starts from a point in the line which is 5 metre from the nearest potato. Find an expression for the total distance run in collecting the potatoes, one at a time and bringing them back one at a time to the starting point. Calculate the value of n if the total distance run is 162 metres.

6. If the sum of first n terms of a sequence be $an^2 + bn$, prove that the sequence is an A. P. and find its common difference ?

6.4 ARITHMETIC MEAN (A. M.)

When three numbers a, A and b are in A. P., then A is called the arithmetic mean of numbers a and b. We have, $A - a = b - A$

or, $$A = \frac{a+b}{2}$$

Thus, the required A. M. of two numbers a and b is $\frac{a+b}{2}$. Consider the following A. P :

3, 8, 13, 18, 23, 28, 33.

There are five terms between the first term 3 and the last term 33. These terms are called *arithmetic means* between 3 and 33. Consider another A. P. : 3, 13, 23, 33. In this case there are two arithmetic means 13, and 23 between 3 and 33.

Generally any number of arithmetic means can be inserted between any two numbers a and b. Let $A_1, A_2, A_3, ..., A_n$ be n arithmetic means between a and b, then.

$a, A_1, A_2, A_3 ..., A_n, b$ is an A. P.

Let d be the common difference of this A. P. Clearly it contains $(n + 2)$ terms

$\therefore \quad b = (n + 2)^{th}$ term

$= a + (n + 1)\, d$

$$\therefore \quad d = \frac{b-a}{n+1}$$

Now, $A_1 = a + d \Rightarrow A_1 = \left(a + \frac{b-a}{n+1}\right)$ ...(i)

$$A_2 = a + 2d \Rightarrow A_2 = \left[a + \frac{2(b-a)}{n+1} \right] \qquad \text{...(ii)}$$

$$\vdots$$

$$A_n = a + nd \Rightarrow An \left[a + \frac{n(b-a)}{n+1} \right] \qquad \text{... }(n)$$

These are required n arithmetic means between a and b.

Adding (i), (ii), ..., (n), we get

$$A_1 + A_2 + \ldots + A_n = na + \cdots + \frac{b-a}{n+1}[1+2+\ldots n]$$

$$= na + \left(\frac{b-a}{n+1}\right)\left(\frac{n(n+1)}{2}\right) = na + \frac{n(b-a)}{2} = \frac{n(a+b)}{2}$$

$= n$ [Single A. M. between a and b]

Example 6.10 Insert five arithmetic means between 8 and 26.

Solution : Let A_1, A_2, A_3, A_4 and A_5 be five arithmetic means between 8 and 26.

Therefore, 8, A_1, A_2, A_3, A_4, A_5, 26 are in A. P. with $a = 8$, $b = 26$, $n = 7$

We have $26 = 8 + (7 - 1)\,d \quad \therefore \quad d = 3$

$\therefore \quad A_1 = a + d = 8 + 3 = 11$, $A_2 = a + 2d = 8 + 2 \times 3 = 14$

$A_3 = a + 3d = 17$, $A_4 = a + 4d = 20$, $A_5 = a + 5d = 23$

Hence, the five arithmetic means between 8 and 26 are 11, 14, 17, 20 and 23.

Example 6.11 The 'n', A. M's between 20 and 80 are such that the ratio of the first mean and the last mean is 1 : 3. Find the value of n.

Solution : Here, 80 is the $(n+2)^{th}$ term of the A. P., whose first term is 20. Let d be the common difference.

$\therefore \quad 80 = 20 + (n+2-1)\,d \quad$ or, $\quad 80 - 20 = (n+1)\,d \quad$ or, $\quad d = \dfrac{60}{n+1}$

The first A. M. $= 20 + \dfrac{60}{n+1} = \dfrac{20n + 20 + 60}{n+1} = \dfrac{20n+80}{n+1}$

The last A. M. $= 20 + n \times \dfrac{60}{n+1} = \dfrac{80n+20}{n+1}$

We have $\frac{20n+80}{n+1} : \frac{80n+20}{n+1} = 1 : 3$ or, $\frac{n+4}{4n+1} = \frac{1}{3}$

or, $4n + 1 = 3n + 12$ or, $n = 11$

$\therefore$ The number of A. M's between 20 and 80 is 11.

CHECK YOUR PROGRESS 6.3

1. Prove that if the number of terms of an A. P. is odd then the middle term is the A. M. between the first and last terms.

2. Between 7 and 85, m number of arithmetic means are inserted so that the ratio of $(m-3)^{th}$ and m^{th} means is 11 : 24. Find the value of m.

3. Prove that the sum of n arithmetic means between two numbers is n times the single A. M. between them.

4. If the A. M. between p^{th} and q^{th} terms of an A. P., be equal and to the A. M. between r^{th} and s^{th} terms of the A. P., then show that $p + q = r + s$.

6.5 GEOMETRIC PROGRESSION

Let us consider the following sequence of numbers :

(1) $1, 2, 4, 8, 16, \cdots$ (2) $3, 1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27} \cdots$

(3) $1, -3, 9, -27, \cdots$ (4) $x, x^2, x^3, x^4, \cdots$

If we see the patterns of the terms of every sequence in the above examples each term is related to the leading term by a definite rule.

For Example (1), the first term is 1, the second term is twice the first term, the third term is 2^2 times of the leading term.

Again for Example (2), the first term is 3, the second term is $\frac{1}{3}$ times of the first term, third term is $\frac{1}{3^2}$ times of the first term.

A sequence with this property is called a *gemetric progression.*

A sequence of numbers in which the ratio of any term to the term which immediately precedes is the same non zero number (other than1), is called a geometric progression or simply G. P. This ratio is called the common ratio.

Thus, $\frac{\textit{Second term}}{\textit{First term}} = \frac{\textit{Third term}}{\textit{Second term}} =$ ..... is called the common ratio of the geometric progression.

Examples (1) to (4) are geometric progressions with the first term 1, 3, 1, x and with common ratio $2, \frac{1}{3}, -3$, and x respectively.

The most general form of a G. P. with the first term a and common ratio r is $a, ar, ar^2, ar^3, \ldots$

6.5.1 GENERAL TERM

Let us consider a geometric progression with the first term a and common ratio r. Then its terms are given by $a, ar, ar^2, ar^3, \ldots$

In this case, $t_1 = a = ar^{1-1}$ $\quad t_2 = ar = ar^{2-1}$

$$t_3 = ar^2 = ar^{3-1} \qquad t_4 = ar^3 = ar^{4-1}$$

$$\ldots \qquad \ldots$$

On generalisation, we get the expression for the n^{th} term as $t_n = ar^{n-1}$ ... (A)

6.5.2 SOME PROPERTIES OF G. P.

(i) If all the terms of a G. P. are multiplied by the same non-zero quantity, the resulting series is also in G. P. The resulting G. P. has the same common ratio as the original one.

If $a, b, c, d, \ldots$ are in G. P.

then $ak, bk, ck, dk \ldots$ are also in G. P. $\quad (k \neq 0)$

(ii) If all the terms of a G. P. are raised to the same power, the resulting series is also in G. P.

Let $a, b, c, d \ldots$ are in G. P.

the $a^k, b^k, c^k, d^k, \ldots$ are also in G. P. $\quad (k \neq 0)$

The common ratio of the resulting G. P. will be obtained by raising the same power to the original common ratio.

Example 6.12 Find the 6^{th} term of the G. P.: 4, 8, 16, ...

Solution : In this case the first term $(a) = 4$ Common ratio $(r) = 8 \div 4 = 2$

Now using the formula $t_n = ar^{n-1}$, we get $t_6 = 4 \times 2^{6-1} = 4 \times 32 = 128$

Hence, the 6^{th} term of the G. P. is 128.

Example 6.13 The 4^{th} and the 9^{th} term of a G. P. are 8 and 256 respectively. Find the G. P.

Solution : Let a be the first term and r be the common ratio of the G. P., then

$$t_4 = ar^{4-1} = ar^{3}, \; t_9 = ar^{9-1} = ar^8$$

According to the question, $ar^8 = 256$ $\quad \cdots (1)$

and $\quad ar^3 = 8$ $\quad \cdots (2)$

$$\therefore \quad \frac{ar^8}{ar^3} = \frac{256}{8} \text{ or } \quad r^5 = 32 = 2^5 \quad \therefore\ r = 2$$

Again from (2), $a \times 2^3 = 8 \quad \therefore \quad a = \frac{8}{8} = 1$

Therefore, the G. P. is 1, 2, 4, 8, 16, ...

Example 6.14 Which term of the G. P.: 5, –10, 20, – 40, ... is 320?

Solution : In this case, $a = 5$; $r = \frac{-10}{5} = -2$.

Suppose that 320 is the n^{th} term of the G. P.

By the formula, $t_n = ar^{n-1}$, we get $t_n = 5.\ (-2)^{n-1}$

$\therefore \quad 5.\ (-2)^{n-1} = 320 \qquad$ (Given)

$\therefore \quad (-2)^{n-1} = 64 = (-2)^6$

$\therefore \quad n - 1 = 6 \ \therefore \quad n = 7$ Hence, 320 is the 7^{th} term of the G. P.

Example 6.15 If $a, b, c,$ and d are in G. P., then show that $(a + b)^2$, $(b + c)^2$, and $(c + d)^2$ are also in G. P.

Solution. Since $a, b, c,$ and d are in G. P., $\therefore \quad \frac{b}{a} = \frac{c}{b} = \frac{d}{c}$

$\therefore \quad b^2 = ac,\ c^2 = bd,\ ad = bc \qquad$ ...(1)

$$\text{Now, } (a+b)^2 (c+d)^2 = \left[(a+b)(c+d)\right]^2 = (ac + bc + ad + bd)^2$$

$$= (b^2 + c^2 + 2bc)^2 \qquad \text{...[Using (1)]}$$

$$= \left[(b+c)^2\right]^2$$

$\therefore \quad \frac{(c+d)^2}{(b+c)^2} = \frac{(b+c)^2}{(a+b)^2}$ Thus, $(a + b)^2$, $(b + c)^2$, $(c + d)^2$ are in G. P.

CHECK YOUR PROGRESS 6.4

1. The first term and the common ratio of a G. P. are respectively 3 and $-\frac{1}{2}$. Write down the first five terms.

MODULE - II
Sequences And Series

Notes

2. Which term of the G. P. 1, 2, 4, 8, 16, ... is 1024? Is 520 a term of the G. P.?

3. Three numbers are in G. P. Their sum is 43 and their product is 216. Find the numbers in proper order.

4. The n^{th} term of a G. P. is 2×3^n for all n. Find (a) the first term (b) the common ratio of the G. P.

6.6 SUM OF *n* TERMS OF A G. P.

Let a denote the first term and r the common ratio of a G. P. Let S_n represent the sum of first n terms of the G. P. Thus, $S_n = a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1}$ ... (1)

Multiplying (1) by r, we get $r\,S_n = ar + ar^2 + \ldots. + ar^{n-2} + ar^{n-1} + ar^n$ ... (2)

$(1) - (2) \Rightarrow S_n - rS_n = a - ar^n$ or $S_n\,(1 - r) = a\,(1 - r^n)$

$$\therefore \quad S_n = \frac{a(1-r^n)}{1-r} \qquad \ldots.(A)$$

$$= \frac{a(r^n-1)}{r-1} \qquad \ldots.(B)$$

Either (A) or (B) gives the sum up to the n^{th} term when $r \neq 1$. It is convenient to use formula (A) when $|\,r\,| < 1$ and (B) when $|\,r\,| > 1$.

Example 6.16 Find the sum of the G. P.: 1, 3, 9, 27, ... up to the 10^{th} term.

Solution : Here the first term (a) = 1 and the common ratio $(r) = \frac{3}{1} = 3$

Now using the formula, $S_n = \frac{a\left(r^n-1\right)}{r-1}$, $(\because\ r > 1)$ we get $S_{10} = \frac{1.(3^{10}-1)}{3-1} = \frac{3^{10}-1}{2}$

Example 6.17 Find the sum of the G. P.: $\frac{1}{\sqrt{3}}, 1, \sqrt{3}, \ , \ldots, 81$

Solution : Here, $a = \frac{1}{\sqrt{3}}; r = \sqrt{3}$ and $t_n = l = 81$

Now $t_n = 81 = \frac{1}{\sqrt{3}}(\sqrt{3})^{n-1} = (\sqrt{3})^{n-2}$

$$\therefore \quad (\sqrt{3})^{n-2} = 3^4 = (\sqrt{3})^8 \quad \therefore\ n - 2 = 8 \text{ or } n = 10$$

$$\therefore \quad S_n = \frac{\frac{1}{\sqrt{3}}\left[\sqrt{3}^{10} - 1\right]}{\sqrt{3}-1} = \frac{\left(\sqrt{3}\right)^{10} - 1}{3-\sqrt{3}}$$

Example 6.18 Find the sum of the G. P.: 0.6, 0.06, 0.006, 0.0006, $\cdots$ to n terms.

Solution. Here, $a = 0.6 = \frac{6}{10}$ and $r = \frac{0.06}{0.6} = \frac{1}{10}$

Using the formula $S_n = \frac{a(1-r^n)}{1-r}$, we have $\quad [\because r < 1]$

$$S_n = \frac{\frac{6}{10}\left\{1 - \left(\frac{1}{10}\right)^n\right\}}{1 - \frac{1}{10}} = \frac{6}{9}\left(1 - \frac{1}{10^n}\right) = \frac{2}{3}\left(1 - \frac{1}{10^n}\right)$$

Hence, the required sum is $\frac{2}{3}\left(1 - \frac{1}{10^n}\right)$.

Example 6.19 How many terms of the G. P.: 64, 32, 16, $\cdots$ has the sum $127\frac{1}{2}$?

Solution : Here, $a = 64,\ r = \frac{32}{64} = \frac{1}{2}$ (<1) and $S_n = 127\frac{1}{2} = \frac{255}{2}$.

Using the formula $S_n = \frac{a(1-r^n)}{1-r}$, we get

$$S_n = \frac{64\left[1 - \left(\frac{1}{2}\right)^n\right]}{1 - \frac{1}{2}} \quad \Rightarrow \quad \frac{64\left[1 - \left(\frac{1}{2}\right)^n\right]}{1 - \frac{1}{2}} = \frac{255}{2} \quad \text{... (given)}$$

or $\quad 128\left[1 - \left(\frac{1}{2}\right)^n\right] = \frac{255}{2}$ or $1 - \left(\frac{1}{2}\right)^n = \frac{255}{256}$

or $\quad \left(\frac{1}{2}\right)^n = 1 - \frac{255}{256} = \frac{1}{256} = \left(\frac{1}{2}\right)^8 \qquad \therefore \qquad n = 8$

Thus, the required number of terms is 8.

Example 6.20 Find the sum of the following sequence :

2, 22, 222, ... to n terms.

Solution : Let S denote the sum. Then

$$S = 2 + 22 + 222 + \cdots \text{ to } n \text{ terms} = 2\,(1 + 11 + 111 + \cdots \text{ to } n \text{ terms})$$

$$= \frac{2}{9}\,(9 + 99 + 999 + \cdots \text{ to } n \text{ terms})$$

$$= \frac{2}{9}\left\{(10-1)+\left(10^2-1\right)+\left(10^3-1\right)+\quad \ldots to\ n\ terms)\right\}$$

$$= \frac{2}{9}\left\{\left(10-10^2+10^3+\ldots to\ n\ terms\right)-\left(1+1+1+\quad \ldots to\ n\ terms\right)\right\}$$

$$= \frac{2}{9}\left\{\frac{(10^n-1)}{10-1}-n\right\} \qquad [\because 10-10^2+10^3+\cdots \text{ is a G P with } r = -10<1]$$

$$= \frac{2}{9}\left\{\frac{10^n-1-9n}{9}\right\} = \frac{2}{81}\left(10^n-1-9n\right)$$

Example 6. 21 Find the sum up to n terms of the sequence:

$$0.7,\ 0.77,\ 0.777,\ \cdots$$

Solution : Let S denote the sum, then

$$S = 0.7 + 0.77 + 0.777 + \cdots \text{ to } n \text{ terms}$$

$$= 7(0.1 + 0.11 + 0.111 + \cdots \text{ to } n \text{ terms})$$

$$= \frac{7}{9}\,(0.9 + 0.99 + 0.999 + \cdots \text{ to } n \text{ terms})$$

$$= \frac{7}{9}\,\{(1–0.1) + (1–0.01) + (1 – 0.001) + \cdots \text{ to } n \text{ terms}\}$$

$$= \frac{7}{9}\,\{(1 + 1 + 1 + \ldots n \text{ terms}) – (0.1 + 0.01 + 0.001 + \cdots \text{ to } n \text{ terms})\}$$

$$= \frac{7}{9}\left\{n-\left(\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\cdots to\ n\ terms\right)\right\}$$

$$=\frac{7}{9}\left\{n-\frac{\frac{1}{10}\left(1-\frac{1}{10^n}\right)}{1-\frac{1}{10}}\right\} \qquad \text{(Since r < 1)}$$

$$=\frac{7}{9}\left\{n-\frac{1}{9}\left[1-\frac{1}{10^n}\right]\right\}=\frac{7}{9}\left\{\frac{9n-1+10^{-n}}{9}\right\}=\frac{7}{81}\left[9n-1+10^{-n}\right]$$

CHECK YOUR PROGRESS 6.5

1. Find the sum of each of the following G. P's :

   (a) 6, 12, 24, ... to 10 terms (b) $1,-\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16}-$ ... to 20 terms.

2. How many terms of the G. P. 8, 16, 32, 64, $\cdots$ have their sum 8184 ?

3. Show that the sum of the G. P. $a+b+\ldots+l$ is $\dfrac{bl-a^2}{b-a}$

4. Find the sum of each of the following sequences up to n terms.

   (a) 8, 88, 888, ... (b) 0.2, 0.22, 0.222, ...

6.7 INFINITE GEOMETRIC PROGRESSION

So far, we have found the sum of a finite number of terms of a G. P. We will now learn to find out the sum of infinitely many terms of a G P such as. $1,\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},\cdots$

We will proceed as follows: Here $a=1,\ r=\frac{1}{2}$.

The n^{th} term of the G. P. is $t_n=\dfrac{1}{2^{n-1}}$ and sum to n terms

i.e., $$S_n=\frac{1-\frac{1}{2^n}}{1-\frac{1}{2}}=2\left(1-\frac{1}{2^n}\right)=2-\frac{1}{2^{n-1}}<2.$$

So, no matter, how large n may be, the sum of n terms is never more than 2.

So, if we take the sum of all the infinitely many terms, we shall not get more than 2 as answer.

Also note that the recurring decimal 0.3 is really 0.3 + 0.03 + 0.003 + 0.0003 + ...

i.e., 0.3 is actually the sum of the above infinite sequence.

On the other hand it is at once obvious that if we sum infinitely many terms of the G. P. 1, 2, 4, 8, 16, ... we shall get a infiinite sum.

So, sometimes we may be able to add the infinitely many terms of G. P. and sometimes we may not. We shall discuss this question now.

6.7.1 SUM OF INFINITE TERMS OF A G. P.

Let us consider a G. P. with infinite number of terms and common ratio *r*.

Case 1 : We assume that $|r| > 1$

The expression for the sum of *n* terms of the G. P. is then given by

$$S_n = \frac{a(r^n - 1)}{r - 1} = \frac{a\,r^n}{r-1} - \frac{a}{r-1} \qquad \text{... (A)}$$

Now as *n* becomes larger and larger r^n also becomes larger and larger. Thus, when *n* is infinitely large and $|r| > 1$ then the sum is also infinitely large which has no importance in Mathematics. We now consider the other possibility.

Case 2 : Let $|r| < 1$

Formula (A) can be written as $S = \dfrac{a\,(1 - r^n)}{1 - r} = \dfrac{a}{1-r} - \dfrac{ar^n}{1-r}$

Now as *n* becomes infinitely large, r^n becomes infinitely small, i.e., as $n \to \infty$, $r^n \to 0$, then the above expression for sum takes the form $S = \dfrac{a}{1-r}$

Hence, the sum of an infinite G. P. with the first term *a* and common ratio *r* is given by

$$S = \frac{a}{1-r}, \text{ when } |r| < 1 \qquad \text{...(i)}$$

Example 6.22 Find the sum of the infinite G. P. $\dfrac{1}{3}, -\dfrac{2}{9}, \dfrac{4}{27}, -\dfrac{8}{81}, \cdots$

Solution : Here, the first term of the infinite G. P. is $a = \dfrac{1}{3}$, and $r = \dfrac{-\frac{2}{9}}{\frac{1}{3}} = -\dfrac{2}{3}$.

Here, $|r| = \left|-\dfrac{2}{3}\right| = \dfrac{2}{3} < 1$

$\therefore$ Using the formula for sum $S = \dfrac{a}{1-r}$ we have $S = \dfrac{\frac{1}{3}}{1-\left(\frac{-2}{3}\right)} = \dfrac{\frac{1}{3}}{1+\frac{2}{3}} = \dfrac{1}{5}$

Hence, the sum of the given G. P. is $\dfrac{1}{5}$.

Example 6.23 Express the recurring decimal $0.\bar{3}$ as an infinite G. P. and find its value in rational form.

Solution. $0.\bar{3}$ $= 0.3333333\ ...$

$= 0.3 + 0.03 + 0.003 + 0.0003 + ...$

$= \dfrac{3}{10} + \dfrac{3}{10^2} + \dfrac{3}{10^3} + \dfrac{3}{10^4} + \ ...$

The above is an infinite G. P. with the first term $a = \dfrac{3}{10}$ and $r = \dfrac{\frac{3}{10^2}}{\frac{3}{10}} = \dfrac{1}{10} < 1$

Hence, by using the formula $S = \dfrac{a}{1-r}$, we get $0.\bar{3} = \dfrac{\frac{3}{10}}{1-\frac{1}{10}} = \dfrac{\frac{3}{10}}{\frac{9}{10}} = \dfrac{3}{9} = \dfrac{1}{3}$

Hence, the recurring decimal $0.\bar{3} = \dfrac{1}{3}$.

Example 6.24 The distance travelled (in cm) by a simple pendulum in consecutive seconds are 16, 12, 9, ... How much distance will it travel before coming to rest ?

Solution : The distance travelled by the pendulum in consecutive seconds are, 16, 12, 9, ... is an infinite geometric progression with the first term $a = 16$ and r $= \dfrac{12}{16} = \dfrac{3}{4} < 1$.

Hence, using the formula $S = \dfrac{a}{1-r}$ we have

$S = \dfrac{16}{1-\frac{3}{4}} = \dfrac{16}{\frac{1}{4}} = 64$ $\therefore$ Distance travelled by the pendulum is 64 cm.

Example 6.25 The sum of an infinite G. P. is 3 and sum of its first two terms is $\frac{8}{3}$. Find the first term.

Solution: In this problem $S = 3$. Let a be the first term and r be the common ratio of the given infinite G. P.

Then according to the question. $a + ar = \frac{8}{3}$

or, $\quad 3a(1+r) = 8 \qquad \text{... (1)}$

Also from $S = \frac{a}{1-r}$, we have $3 = \frac{a}{1-r}$

or, $\quad a = 3(1-r) \qquad \text{... (2)}$

From (1) and (2), we get.

$3.3\,(1-r)\,(1+r) = 8$

or, $\quad 1 - r^2 = \frac{8}{9}$ or, $\quad r^2 = \frac{1}{9}$

or, $\quad r = \pm\frac{1}{3}$

From (2), $a = 3\left(1 \mp \frac{1}{3}\right) = 2$ or 4 according as $r = \pm\frac{1}{3}$.

CHECK YOUR PROGRESS 6.6

(1) Find the sum of each of the following inifinite G. P's :

(a) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots \infty$ (b) $\frac{2}{5} + \frac{3}{5^2} + \frac{2}{5^3} + \frac{3}{5^4} + \cdots \infty$

2. Express the following recurring decimals as an infinite G. P. and then find out their values as a rational number. (a) $0.\overline{7}$ (b) $0.3\overline{15}$

3. The sum of an infinite G. P. is 15 and the sum of the squares of the terms is 45. Find the G.P.

4. The sum of an infinite G. P. is $\frac{1}{3}$ and the first term is $\frac{1}{4}$. Find the G.P.

6.8 GEOMETRIC MEAN (G. M.)

If a, G, b are in G. P., then G is called the geometric mean between a and b.

If three numbers are in G. P., the middle one is called the geometric mean between the other two.

If $a, G_1, G_2, \ldots, G_n, b$ are in G. P.,

then $G_1, G_2, \ldots G_n$ are called n G. M.'s between a and b.

The geometric mean of n numbers is defined as the n^{th} root of their product.

Thus if $a_1, a_2, \ldots, a_n$ are n numbers, then their

$$\text{G. M.} = (a_1, a_2, \ldots a_n)^{\frac{1}{n}}$$

Let G be the G. M. between a and b, then a, G, b are in G. P $\therefore \frac{G}{a} = \frac{b}{G}$

or, $G^2 = ab$ or, $G = \sqrt{ab}$

$\therefore$ Geometric mean $= \sqrt{\text{Product of extremes}}$

Given any two positive numbers a and b, any number of geometric means can be inserted between them Let $a_1, a_2, a_3 \ldots, a_n$ be n geometric means between a and b.

Then $a_1, a_1, a_3, \ldots\ldots. a_n, b$ is a G. P.

Thus, b being the $(n + 2)^{th}$ term, we have

$$b = a\, r^{n+1}$$

or, $r^{n+1} = \frac{b}{a}$ or, $r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

Hence, $a_1 = ar = a \times \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$, $a_2 = ar^2 = a \times \left(\frac{b}{a}\right)^{\frac{2}{n+1}}$

$\ldots \qquad \ldots$

$\ldots \qquad \ldots$

$$a_n = ar^n = a \times \left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

Further we can show that the product of these n G. M.'s is equal to n^{th} power of the single geometric mean between a and b.

Multiplying $a_1 . a_2, \ldots a_n$, we have

$$a_1, a_2 \cdots a_n = a^n \left(\frac{b}{a}\right)^{\frac{1}{n+1}+\frac{2}{n+1}+\cdots+\frac{n}{n+1}} = a^n \left(\frac{b}{a}\right)^{\frac{1+2+\cdots+n}{n+1}} = a^n \left(\frac{b}{a}\right)^{\frac{n(n+1)}{2(n+1)}}$$

$$= a^n \left(\frac{b}{a}\right)^{\frac{n}{2}} = (ab)^{\frac{n}{2}} = \left(\sqrt{ab}\right)^n = \text{G}^n = (\text{single G. M. between } a \text{ and } b)^n$$

Example 6.26 Find the G. M. between $\frac{3}{2}$ and $\frac{27}{2}$

Solution : We know that if a is the G. M. between a and b, then $G = \sqrt{ab}$

$\therefore$ G. M. between $\frac{3}{2}$ and $\frac{27}{2} = \sqrt{\frac{3}{2} \times \frac{27}{2}} = \frac{9}{2}$

Example 6.27 Insert three geometric means between 1 and 256.

Solution : Let G_1, G_2, G_3, be the three geometric means between 1 and 256.

Then 1, G_1, G_2, G_3, 256 are in G. P.

If r be the common ratio, then $t_5 = 256$ i.e, $ar^4 = 256 \Rightarrow 1.\ r^4 = 256$

or, $r^2 = 16$ or, $r = \pm 4$

When $r = 4$, $G_1 = 1.\ 4 = 4$, $G_2 = 1.\ (4)^2 = 16$ and $G_3 = 1.\ (4)^3 = 64$

When $r = -4$, $G_1 = -4$, $G_2 = (1)\ (-4)^2 = 16$ and $G_3 = (1)\ (-4)^3 = -64$

$\therefore$ G.M. between 1 and 256 are 4, 16, 64, or, – 4, 16, –64.

Example 6.28 If 4, 36, 324 are in G. P. insert two more numbers in this progression so that it again forms a G. P.

Solution : G. M. between 4 and 36 = $\sqrt{4 \times 36} = \sqrt{144} = 12$

G. M. between 36 and $324 = \sqrt{36 \times 324} = 6 \times 18 = 108$

If we introduce 12 between 4 and 36 and 108 betwen 36 and 324, the numbers 4, 12, 36, 108, 324 form *a* G. P.

$\therefore$ The two new numbers inserted are 12 and 108.

Example 6.29 Find the value of n such that $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ may be the geometric mean between a and b.

Solution : If x be G. M. between a and b, then $x = a^{\frac{1}{2}} \times b^{\frac{1}{2}}$

$$\therefore \quad \frac{a^{n+1}+b^{n+1}}{a^n+b^n} = a^{\frac{1}{2}}\, b^{\frac{1}{2}} \text{ or, } \quad a^{n+1}+b^{n+1} = \left(a^{\frac{1}{2}}\, b^{\frac{1}{2}}\right)\left(a^n+b^n\right)$$

$$\text{or, } \quad a^{n+1}+b^{n+1} = a^{n+\frac{1}{2}}\, b^{\frac{1}{2}} + a^{\frac{1}{2}}\, b^{n+\frac{1}{2}} \text{ or, } a^{n+1} - a^{n+\frac{1}{2}}.\, b^{\frac{1}{2}} = a^{\frac{1}{2}}\, b^{n+\frac{1}{2}} - b^{n+1}$$

$$\text{or, } \quad a^{n+\frac{1}{2}}\left(a^{\frac{1}{2}} - b^{\frac{1}{2}}\right) = b^{n+\frac{1}{2}}\left(a^{\frac{1}{2}} - b^{\frac{1}{2}}\right) \text{ or, } \quad a^{n+\frac{1}{2}} = b^{n+\frac{1}{2}}$$

$$\text{or, } \quad \frac{a^{n+\frac{1}{2}}}{b^{n+\frac{1}{2}}} = 1 \text{ or, } \quad \left(\frac{a}{b}\right)^{n+\frac{1}{2}} = \left(\frac{a}{b}\right)^{0}$$

$$\therefore \quad n + \frac{1}{2} = 0 \text{ or, } n = \frac{-1}{2}$$

6.8.1 RELATIONSHIP BETWEEN A. M. AND G.M.

Let a and b be the two numbers.

Let A and G be the A. M. and G. M. respectively between a and b

$$\therefore \quad A = \frac{a+b}{2}, G = \sqrt{ab}$$

$$A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{\left(\sqrt{a}\right)^2 + \left(\sqrt{b}\right)^2 - 2\sqrt{ab}}{2} = \frac{1}{2}\left(\sqrt{a} - \sqrt{b}\right)^2 > 0$$

$$\therefore \quad A > G$$

Example 6.30 The arithemetic mean between two numbers is 34 and their geometric mean is 16. Find the numbers.

Solution : Let the numbers be a and b. Since A. M. between a and b is 34,

$$\therefore \quad \frac{a+b}{2} = 34, \text{ or, } a + b = 68 \qquad \ldots (1)$$

Since G. M. between a and b is 16,

$$\therefore \quad \sqrt{ab} = 16 \text{ or, } ab = 256 \text{ we know that } (a-b)^2 = (a+b)^2 - 4\,ab \qquad \cdots (2)$$

$$= (68)^2 - 4 \times 256 = 4624 - 1024 = 3600$$

$\therefore \quad a - b = \sqrt{3600} = 60 \qquad \cdots(3)$

Adding (1) and (3), we get, $2a = 128 \quad \therefore \quad a = 64$

Subtracting (3) from (1), we get

$2b = 8$ or, $b = 4$

$\therefore$ Required numbers are 64 and 4.

Example 6.31 The arithmetic mean between two quantities b and c is a and the two geometric means between them are g_1 and g_2. Prove that $g_1^3 + g_2^3 = 2\,abc$

Solution : The A. M. between b and c is a $\therefore \quad \dfrac{b+c}{2} = a$, or, $b + c = 2a$

Again g_1 and g_2 are two G. M.'s between b and c $\therefore \quad b, g_1, g_2, c$ are in G. P.

If r be the common ratio, then $c = br^3$ or, $r = \left(\dfrac{c}{b}\right)^{\frac{1}{3}}$

$$g_1 = br = b\left(\frac{c}{b}\right)^{\frac{1}{3}} \quad \text{and} \quad g_2 = br^2 = b\left(\frac{c}{b}\right)^{\frac{2}{3}}$$

$$\therefore \quad g_1^3 + g_2^3 = b^3\left[\left(\frac{c}{b}\right) + \left(\frac{c}{b}\right)^2\right] = b^3 \times \frac{c}{b}\left(1 + \frac{c}{b}\right) = b^2 c \times \left(\frac{b+c}{b}\right)$$

$= bc\,(2a) \qquad$ [since $b + c = 2a$]

$= 2abc$

Example 6.32 The product of first three terms of a G. P. is 1000. If we add 6 to its second term and 7 to its 3rd term, the three terms form an A. P. Find the terms of the G. P.

Solution : Let $t_1 = \dfrac{a}{r}, t_2 = a$ and $t_3 = ar$ be the first three terms of G. P.

Then, their product $= \dfrac{a}{r}.\, a.\, ar = 1000$ or, $a^3 = 1000$, or, $a = 10$

By the question, t_1, $t_2 + 6$, $t_3 + 7$ are in A. P. $\qquad$ ...(1)

i.e. $\frac{a}{r}$, $a + 6$, $ar + 7$ are in A. P.

$\therefore \quad (a+6) - \frac{a}{r} = (ar+7) - (a+6)$ or, $\quad 2(a+6) = \frac{a}{r} + (ar+7)$

or, $\quad 2(10+6) = \frac{10}{r} + (10r+7) \qquad$ [using (1)]

or, $\quad 32r = 10 + 10r^2 + 7r \quad$ or, $\quad 10r^2 - 25r + 10 = 0$

$$\therefore \quad r = \frac{25 \pm \sqrt{625-400}}{20} = \frac{25 \pm 15}{20} = 2, \frac{1}{2}$$

When $a = 10$, $r = 2$. then the terms are $\frac{10}{2}$, $10(2)$ i.e., 5, 10, 20

When $a = 10$, $r = \frac{1}{2}$ then the terms are $10(2)$, 10, $10\left(\frac{1}{2}\right)$ i.e., 20, 10, 5

CHECK YOUR PROGRESS 6.7

1. Insert 8 G. M.'s between 8 and $\frac{1}{64}$.

2. If a_1 is the first of n geometric means between a and b, show that $a_1^{n+1} = a^n b$

3. If G is the G. M. between a and b, prove that $\frac{1}{G^2 - a^2} + \frac{1}{G^2 - b^2} = \frac{1}{G^2}$

4. If the A. M. and G. M. between two numbers are in the ratio $m : n$, then prove that the numbers are in the ratio $m + \sqrt{m^2 - n^2} : m - \sqrt{m^2 - n^2}$

5. If A and G are respectvely arithmetic and geometric means between two numbers a and b, then show that A > G.

6. The sum of first three terms of a G. P. is $\frac{13}{12}$ and their product is –1. Find the G. P.

7. The product of three terms of a G. P. is 512. If 8 is added to first and 6 is added to second term, the numbers form an A. P., Find the numbers.

LET US SUM UP

- A sequence in which the difference of two cousecutive terms is always constant ($\neq 0$) is called an Arithmetic Progression (A. P.)

- The general term of an A. P.

  $a, a + d, a + 2d, \ldots$ is given by $t_n = a + (n - 1)\, d$

- $S_{n,}$ the sum of the first n terms of the A.P a, a+d, a+2d,... is given by

  $$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

  $$= \frac{n}{2}\,(a + l), \text{ where } l = a + (n - 1)\, d.$$

- $t_n = S_n - S_{n-1}$

- An arithmetic mean between a and b is $\dfrac{a+b}{2}$.

- A sequence in which the ratio of two consecutive terms is always constant ($\neq 0$) is called a Geometric Progression (G. P.)

- The n^{th} term of a G. P.: $a, ar, ar^2, \ldots$ is ar^{n-1}

- Sum of the first n terms of a G. P.: $a, ar, ar^2, \ldots$ is

  $$S_n = \frac{a(r^n - 1)}{r - 1} \text{ for } | r | > |$$

  $$= \frac{a(1 - r^n)}{1 - r} \text{ for } | r | < 1$$

- The sums of an infintite G. P. $a, ar, ar^2, \ldots$ is given by

  $$S = \frac{a}{1 - r} \text{ for } | r | < 1$$

- Geometric mean G between two numbers a and b is $\sqrt{ab}$

- The arithmetic mean A between two numbers a and b is always greater than the corresponding Geometric mean G i.e., A > G.

SUPPORTIVE WEB SITES

http://www.youtube.com/watch?v=_cooC3yG_p0

http://www.youtube.com/watch?v=pXo0bG4iAyg

http://www.youtube.com/watch?v=dIGLhLMsy2U

http://www.youtube.com/watch?v=cYw4MFWsB6c

http://www.youtube.com/watch?v=Uy_L8tnihDM

http://www.bbc.co.uk/education/asguru/maths/13pure/03sequences/index.shtml

TERMINAL EXERCISE

1. Find the sum of all the natural numbers between 100 and 200 which are divisible by 7.

2. The sum of the first n terms of two A. P.'s are in the ratio $(2n-1):(2n+1)$. Find the ratio of their 10^{th} terms.

3. If a, b, c are in A. P. then show that $b+c, c+a, a+b$ are also in A. P.

4. If $a_1, a_2, \ldots, a_n$ are in A. P., then prove that

$$\frac{1}{a_1 a_2}+\frac{1}{a_2 a_3}+\frac{1}{a_3 a_4}+\ldots+\frac{1}{a_{n-1} a_n}=\frac{n-1}{a_1 a_n}$$

5. If $(b-c)^2, (c-a)^2, (a-b)^2$ are in A. P., then prove that

$\frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}$, are also in A. P.

6. If the p^{th}, q^{th} and r^{th} terms are P, Q, R respectively. Prove that $P(Q-R)+Q(R-P)+r(P-Q)=0$.

7. If a, b, c are in G. P. then prove that $a^2 b^2 c^2\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)=a^3+b^3+c^3$

8. If a, b, c, d are in G. P., show that each of the following form a G. P. :

(a) $(a^2-b^2), (b^2-c^2), (c^2-d^2)$ (b) $\frac{1}{a^2+b^2}, \frac{1}{b^2+c^2}, \frac{1}{c^2-d^2}$

9. If x, y, z are the p^{th}, q^{th} and r^{th} terms of a G. P., prove that $x^{q-r} y^{r-p} z^{p-q}=1$

10. If a, b, c are in A. P. and x, y, z are in G. P. then prove that $x^{b-c} y^{c-a} z^{a-b}=1$

11. If the sum of the first n terms of a G. P. is represented by S_n, then prove that

$S_n(S_{3n}-S_{2n})=(S_{2n}-S_n)^2$

12. If p, q, r are in A. P. then prove that the p^{th}, q^{th} and r^{th} terms of a G. P. are also in G. P.

13. If $S_n=1+\frac{1}{2}+\frac{1}{2^2}\ \ldots+\frac{1}{2^{n-1}}$, find the least value of n such that

$$2-S_n<\frac{1}{100}$$

14. If the sum of the first n terms of a G. P. is S and the product of these terms is p and the sum of their reciprocals is R, then prove that $p^2=\left(\frac{S}{R}\right)^n$

ANSWERS

CHECK YOUR PROGRESS 6.1

1. (a) $2n-1$ (b) $2n+1$ 2. 3, 5, 7, 9, ... 3. 20, 16
4. no 5. $m+n$ 6. 10, 6, 2,

CHECK YOUR PROGRESS 6.2

1. (a) 435 (b) $\frac{n}{2}\left[21-5n^2\right]$ 2. 5 3. 12
4. 37 5. $n^2+9n, 9$ 6. $2a$

CHECK YOUR PROGRESS 6. 3

2. 5

CHECK YOUR PROGRESS 6. 4

1. $3, -\frac{3}{2}, \frac{3}{4}, -\frac{3}{8}, \frac{3}{16}$ 2. 11^{th}, no 3. 36, 6, 1 or 1, 6, 36
4. (a) 6 (b) 3

CHECK YOUR PROGRESS 6.5

1. (a) 6138 (b) $\frac{2}{3}\left(1-\frac{1}{2^{20}}\right)$ 2. 10.
4. (a) $\frac{80}{81}\left(10^n-1\right)-\frac{8n}{9}$ (b) $\frac{2n}{9}-\frac{2}{81}\left(1-\frac{1}{10^n}\right)$

CHECK YOUR PROGRESS 6. 6

1. (a) $\frac{3}{2}$ (b) $\frac{13}{24}$ 2. (a) $\frac{7}{9}$ (b) $\frac{52}{165}$
3. $5, \frac{10}{3}, \frac{20}{9}, \frac{40}{27}, \ldots \infty$
4. $\frac{1}{4}, \frac{1}{4^2}, \frac{1}{4^3}, \frac{1}{4^4}, \ldots \infty$

CHECK YOUR PROGRESS 6.7

1. $4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32},$ 6. $\frac{4}{3}, -1, \frac{3}{4}$ ... or $\frac{3}{4}, -1, \frac{4}{3}$ ... 7. 4, 8, 16

TERMINAL EXERCISE

1. 2107 2. 37 : 39

7

SOME SPECIAL SEQUENCES

Suppose you are asked to collect pebbles every day in such a way that on the first day if you collect one pebble, second day you collect double of the pebbles that you have collected on the first day, third day you collect double of the pebbles that you have collected on the second day, and so on. Then you write the number of pebbles collected daywise, you will have a sequence, $1, 2, 2^2, 2^3, ...$

From a sequence we derive a series. The series corresponding to the above sequence is

$1 + 2 + 2^2 + 2^3 + ...$

One well known series is Fibonacci series 1 + 1 + 2 + 3 + 5+ 8 + 13 + ...

In this lesson we shall study some special types of series in detail.

OBJECTIVES

After studying this lesson, you will be able to :

- define a series;
- calculate the terms of a series for given values of n from t_n;
- evaluate $\sum n, \sum n^2, \sum n^3$ using method of differences and mathematical induction; and
- evaluate simple series like 1.3 + 3.5 + 5.7 +..... n terms.

EXPECTED BACKGROUND KNOWLEDGE

- Concept of a sequence
- Concept of A. P. and G. P., sum of n terms.
- Knowldge of converting recurring decimals to fractions by using G. P.

7.1 SERIES

An expression of the form $u_1 + u_2 + u_3 + ... + u_n + ...$ is called a series, where $u_1, u_2, u_3 ..., u_n$ ... is a sequence of numbers. The above series is denoted by $\sum_{r=1}^{n} u_r$. If n is finite then the series is a finite series, otherwise the series is infinite. Thus we find that a series is associated to a sequence. Thus a series is a sum of terms arranged in order, according to some definite law.

Consider the following sets of numbers :

(a) 1, 6, 11, ..., (b) $\frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \frac{1}{12} \cdots$

(c) 48, 24, 12, ..., (d) $1^2, 2^2, 3^2, ...$

(a), (b), (c), (d) form sequences, since they are connected by a definite law. The series associated with them are :

$$1 + 6 + 11 + ..., \frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \frac{1}{12} + ..., 48 + 24 + 12 + ..., 1^2 + 2^2 + 3^2 + ...$$

Example 7.1 Write the first 6 terms of each of the following sequences, whose n^{th} term is given by

(a) $T_n = 2n + 1$, (b) $a_n = n^2 - n + 1$ (c) $f_n = (-1)^n . 5^n$

Hence find the series associated to each of the above sequences.

Solution : (a) $T_n = 2n + 1$, For $n = 1$, $T_1 = 2.1 + 1 = 3$, For $n = 2$, $T_2 = 2.2 + 1 = 5$

For $n = 3$, $T_3 = 2.3 + 1 = 7$, For $n = 4$, $T_4 = 2.4 + 1 = 9$

For $n = 5$, $T_5 = 2.5 + 1 = 11$, For $n = 6$, $T_6 = 2.6 + 1 = 13$

Hence the series associated to the above sequence is $3 + 5 + 7 + 9 + 11 + 13 + ...$

(b) $a_n = n^2 - n + 1$, For $n = 1$, $a_1 = 1^2 - 1 + 1 = 1$

For $n = 2$, $a_2 = 2^2 - 2 + 1 = 3$, For $n = 3$, $a_3 = 3^2 - 3 + 1 = 7$

For $n = 4$, $a_4 = 4^2 - 4 + 1 = 13$,For $n = 5$, $a_5 = 5^2 - 5 + 1 = 21$

For $n = 6$, $a_6 = 6^2 - 6 + 1 = 31$

Hence the series associated to the above sequence is $1 + 3 + 7 + 13 +...$

(c) Here $f_n = (-1)^n 5^n$, For $n = 1$, $f_1 = (-1)^1 5^1 = -5$

For $n = 2$, $f_2 = (-1)^2 5^2 = 25$, For $n = 3$, $f_3 = (-1)^3 5^3 = -125$

For $n = 4$, $f_4 = (-1)^4 5^4 = 625$, For $n = 5$, $f_5 = (-1)^5 5^5 = -3125$

For $n = 6$, $f_6 = (-1)^6 5^6 = 15625$

The corresponding series relative to the sequence

$f_n = (-1)^n 5^n$ is $-5 + 25 - 125 + 625 - 3125 + 15625-$

Example 7.2 Write the n^{th} term of each of the following series :

(a) $-2 + 4 - 6 + 8 - ....$ (b) $1 - 1 + 1 - 1 + ...$

(c) $4 + 16 + 64 + 256 + ...$ (d) $\sqrt{2} + \sqrt{3} + 2 + \sqrt{5} + ...$

Solution : (a) The series is $-2 + 4 - 6 + 8 ....$

Here the odd terms are negative and the even terms are positive. The above series is obtained by multiplying the series. $-1 + 2 - 3 + 4 - ....$ by 2

$\therefore$ $T_n = 2(-1)^n n = (-1)^n 2n$

(b) The series is $1 - 1 + 1 - 1 + 1 - ......$

$\therefore$ $T_n = (-1)^{n+1}$

(c) The series is $4 + 16 + 64 + 256 + ...$

The above series can be writen as $4 + 4^2 + 4^3 + 4^4 + ...$

i.e., n^{th} term, $T_n = 4^n$.

(d) The series is $\sqrt{2} + \sqrt{3} + 2 + \sqrt{5} + ...$ *i.e.,* $\sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{5} + ...$

$\therefore$ n^{th} term is $T_n = \sqrt{n+1}$.

CHECK YOUR PROGRESS 7.1

1. Write the first 6 terms of each of the following series, whose n^{th} term is given by

(a) $T_n = \dfrac{n(n+1)(n+2)}{6}$ (b) $a_n = \dfrac{n^2 - 1}{2n - 3}$

2. If $A_1 = 1$ and $A_2 = 2$, find A_6 if $A_n = \dfrac{A_{n-1}}{A_{n-2}}, (n > 2)$

3. Write the n^{th} term of each of the following series:

(a) $-1 + \dfrac{1}{2} - \dfrac{1}{3} + \dfrac{1}{4} - \cdots$ (b) $3 - 6 + 9 - 12 + ...$

7.2 SUM OF THE POWERS OF THE FIRST *n* NATURAL NUMBERS

(a) The series of first n natural numbers is

$1 + 2 + 3 + 4 + ... + n.$

Let $S_n = 1 + 2 + 3 + ... + n$

This is an arithmetic series whose the first term is 1, the common difference is 1 and the number of terms is n. $\therefore \quad S_n = \frac{n}{2}\left[2.1+(n-1)1\right] = \frac{n}{2}[2n-1]$

i.e., $\quad S_n = \frac{n(n+1)}{2}$

$\therefore \quad$ We can write $\sum n = \frac{n(n+1)}{2}$

(b) Determine the sum of the squares of the first n natural numbers.

Let $\quad S_n = 1^2 + 2^2 + 3^2 + ... + n^2$

Consider the identity : $n^3 - (n-1)^3 = 3n^2 - 3n + 1$

By giving the values for $n = 1, 2, 3, ..., n-1, n$ in the above identity, we have.

$1^3 - 0^3 = 3.1^2 - 3.1 + 1$

$2^3 - 1^3 = 3.2^2 - 3.2 + 1$

$3^3 - 2^3 = 3.3^2 - 3.3 + 1$

....

....

$n^3 - (n-1)^3 = 3n^2 - 3n + 1$

Adding these we get

$n^3 - 0^3 = 3\,(1^2 + 2^2 + 3^2 + ... + n^2) - 3\,(1 + 2 + 3 + ... + n) +$

$(1 + 1 + 1 + ... \; n$ times$)$

or, $\quad n^3 = 3\,S_n - 3\left[\frac{n\,(n+1)}{2}\right] + n \; .... \left[\because \sum n = \frac{n\,(n+1)}{2}\right]$

or, $\quad 3\,S_n = n^3 + \frac{3n\,(n+1)}{2} - n = n\,(n^2 - 1) + \frac{3n}{2}\,(n+1)$

$= n\,(n+1)\left[n - 1 + \frac{3}{2}\right] = \frac{n\,(n+1)\,(2n+1)}{2}$

$\therefore \quad S_n = \frac{n\,(n+1)\,(2n+1)}{6}$ i.e., $\quad \sum n^2 = \frac{n\,(n+1)\,(2n+1)}{6}$

(c) Determine the sum of the cubes of the first n natural numbers.

Here $S_n = 1^3 + 2^3 + 3^3 + ... + n^3$

Consider the identity : $n^4 - (n-1)^4 = 4n^3 - 6n^2 + 4n - 1$

Putting successively 1, 2, 3, ... for n we have

$$1^4 - 0^4 = 4.1^3 - 6.1^2 + 4.1 - 1$$

$$2^4 - 1^4 = 4.2^3 - 6.2^2 + 4.2 - 1$$

$$3^4 - 2^4 = 4.3^3 - 6.3^2 + 4.3 - 1$$

$$\cdots \qquad \cdots$$

$$n^4 - (n-1)^4 = 4.n^3 - 6.n^2 + 4.n - 1$$

Adding these, we get

$$n^4 - 0^4 = 4(1^3 + 2^3 + \ldots + n^3) - 6(1^2 + 2^2 + \ldots + n^2) + 4\,(1 + 2 + 3 + \ldots + n) - (1 + 1 + \ldots n \text{ times})$$

$$\Rightarrow \qquad n^4 = 4.S_n - 6\left\{\frac{n\,(n+1)\,(2n+1)}{6}\right\} + 4n\,\frac{n+1}{2} - n$$

$$\Rightarrow \qquad 4S_n = n^4 + n(n+1)\,(2n+1) - 2n\,(n+1) + n$$

$$= n^4 + n\,(2n^2 + 3n + 1) - 2n^2 - 2n + n$$

$$= n^4 + 2n^3 + 3n^2 + n - 2n^2 - 2n + n = n^4 + 2n^3 + n^2 = n^2\,(n^2 + 2n + 1)$$

i.e., $4S_n = n^2\,(n+1)^2$

$$\therefore \qquad S_n = \frac{n^2\,(n+1)^2}{4} = \left\{\frac{n\,(n+1)}{2}\right\}^2$$

$$\therefore \qquad \sum n^3 = \left[\frac{n(n+1)}{2}\right]^2 \quad \text{or,} \quad \sum n^3 = (\sum n)^2$$

Note : *In problems on finding sum of the series, we shall find the nth term of the series (t_n) and then use $S_n = \sum t_n$.*

Example 7.3 Find the sum of first n terms of the series 1.3 + 3.5 + 5.7 + ...

Solution :

Let $S_n = 1.3 + 3.5 + 5.7 + \ldots$

The n^{th} term of the series

$$t_n = \{n^{th} \text{ term of } 1, 3, 5, \ldots\} \times \{n^{th} \text{ term of } 3, 5, 7, \ldots\}$$

$$= (2n-1)\,(2n+1) = 4n^2 - 1$$

$$S_n = \sum t_n = \sum \left[4n^2 - 1\right]$$

$$= 4\sum n^2 - \sum(1) = 4\,\frac{n\,(n+1)\,(2n+1)}{6} - n$$

$$= \frac{2n\,(n+1)\,(2n+1) - 3n}{3} = \frac{n}{3}\left[2\,(2n^2 + 3n + 1) - 3\right]$$

$$= \frac{n}{3}\left[4n^2 + 6n - 1\right]$$

Example 7.4 Find the sum of first n terms of the series

$$1.2^2 + 2.3^2 + 3.4^2 + \ldots\ldots$$

Solution : Here $t_n = n\,\{2 + (n-1)\}^2 = n\,(n+1)^2 = n.\,(n^2 + 2n + 1)$

i.e., $t_n = n^3 + 2n^2 + n$

Let $S_n = 1.2^2 + 2.3^2 + 2.3^2 + 3.4^2 + \ldots. + n.\,(n+1)^2.$

$\therefore$ $S_n = \sum t_n = \sum\,(n^3 + 2n^2 + n) = \sum n^3 + 2\sum n^2 + \sum n.$

$$= \left\{\frac{n\,(n+1)}{2}\right\}^2 + 2\left\{\frac{n\,(n+1)\,(2n+1)}{6}\right\} + \frac{n\,(n+1)}{2}$$

$$= n\,(n+1)\left\{\frac{n\,(n+1)}{4} + \frac{2n+1}{3} + \frac{1}{2}\right\}$$

$$= \frac{n\,(n+1)}{12}\,(3n^2 + 11n + 10) = \frac{1}{12}\,n\,(n+1)\,(n+2)\,(3n+5)$$

Example 7.5 Find the sum of first n terms of the series

$$2.\,3.\,5 + 3.\,5.\,7 + 4.\,7.\,9 + \ldots.$$

Solution : Let $S_n = 2.\,3.\,5. + 3.\,5.\,7 + 4.\,7.\,9 + \ldots.$

n^{th} term of the series

$t_n =$ {n^{th} term of 2, 3, 4, ...} × {n^{th} term of 3, 5, 7, ...} × {n^{th} term of 5, 7, 9, ...}

$= (n+1) \times (2n+1) \times (2n+3)$

$= (n+1)\,[4n^2 + 8n + 3] = 4n^3 + 12n^2 + 11n + 3$

$\therefore$ $S_n = \sum t_n = \sum\,[4n^3 + 12n^2 + 11n + 3]$

$= 4\sum n^3 + 12\sum n^2 + 11\sum n + \sum(3)$

$$= 4\,\frac{n^2\,(n+1)^2}{4} + \frac{12\,n\,(n+1)\,(2n+1)}{6} + \frac{11\,n\,(n+1)}{2} + 3n$$

$$= n^2\,(n+1)^2 + 2n\,(n+1)\,(2n+1) + \frac{11n\,(n+1)}{2} + 3n$$

$$= \frac{n}{2}\left[2n\,(n+1)^2 + 4\,(n+1)\,(2n+1) + 11\,(n+1) + 6\right]$$

$$= \frac{n}{2}\left[2n\,(n^2+2n+1) + 4\,(2n^2+3n+1) + 11n + 17\right]$$

$$= \frac{n}{2}\left[2n^3 + 12n^2 + 25n + 21\right]$$

Example 7.6 Find the sum of first n terms of the following series :

$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots$$

Solution : $t_n = \dfrac{1}{(2n-1)\,(2n+1)}$

$$= \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$$

Now putting successively for $n = 1, 2, 3, \ldots.$

$$t_1 = \frac{1}{2}\left(1 - \frac{1}{3}\right)$$

$$t_2 = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right)$$

$$t_3 = \frac{1}{2}\left(\frac{1}{5} - \frac{1}{7}\right)$$

$$\cdots \qquad \cdots$$

$$t_n = \frac{1}{2}\left(\frac{1}{(2n-1)} - \frac{1}{(2n+1)}\right)$$

Adding, $t_1 + t_2 + \cdots + t_n = \dfrac{1}{2}\left[1 - \dfrac{1}{2n+1}\right] = \dfrac{n}{(2n+1)}$

CHECK YOUR PROGRESS 7.2

1. Find the sum of first n terms of each of the following series :

   (a) $1 + (1 + 3) + (1 + 3 + 5) + ...$

   (b) $\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \cdots$

   (c) $(1) + (1 + 3) + (1 + 3 + 3^2) + (1 + 3 + 3^2 + 3^3) + ...$

2. Find the sum of n terms of the series. whose nth term is $n(n + 1)(n + 4)$

3. Find the sum of the series $1.\ 2.\ 3 + 2.\ 3.\ 4. + 3.\ 4.\ 5 + \cdots$ upto n terms

LET US SUM UP

- An expression of the form $u_1 + u_2 + u_3 + ... + u_n + ...$ is called a series, where $u_1, u_2, u_3 .... u_n, ...$ is a sequence of numbers
- $$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$
- $$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$
- $$\sum_{r=1}^{n} r^3 = \left\{\frac{n(n+1)}{2}\right\}^2$$
- $S_n = \sum t_n$

SUPPORTIVE WEB SITES

http://en.wikipedia.org/wiki/Sequence_and_series
http://mathworld.wolfram.com/Series.html

TERMINAL EXERCISE

1. Find the sum of each of the following series :

   (a) $2 + 4 + 6 + ...$ up to 40 terms.

(b) $2 + 6 + 18 + \ldots$ up to 6 terms.

2. Sum each of the following series to n terms :

(a) $1 + 3 + 7 + 15 + 31 + \ldots\ldots$

(b) $\frac{1}{1.3.5} + \frac{1}{3.5.7} + \frac{1}{5.7.9} + \ldots\ldots$

(c) $\frac{3}{1.4} + \frac{5}{4.9} + \frac{7}{9.16} + \frac{9}{16.25} + \ldots.$

3. Find the sum of first n terms of the series $1^2 + 3^2 + 5^2 + \ldots.$

4. Find the sum to n terms of the series $5 + 7 + 13 + 31 + \ldots.$

5. Find the sum to n terms of the series $1+\frac{4}{5}+\frac{7}{5^2}+\frac{10}{5^3}+\cdots$

6. Find the sum of $2^2 + 4^2 + 6^2 + \ldots + (2n)^2$

7. Show that

$$\frac{1\times 2^2 + 2\times 3^2 + \ldots\ldots + n\times(n+1)^2}{1^2\times 2 + 2^2\times 3 + \ldots\ldots + n^2\times(n+1)} = \frac{3n+5}{3n+1}$$

ANSWERS

CHECK YOUR PROGRESS 7.1

1. (a) 1, 4, 10, 20, 35, 56 (b) $0, 3, \frac{8}{3}, 3, \frac{24}{7}, \frac{35}{9}$ 2. $\frac{1}{2}$

3. (a) $(-1)^n \frac{1}{n}$ (b) $(-1)^{n+1}\, 3n$

CHECK YOUR PROGRESS 7.2

1. (a) $\frac{1}{6} n\,(n+1)\,(2n+1)$ (b) $\frac{n}{3n+1}$ (c) $\frac{1}{4}(3^{n+1} - 2n - 3)$

2. $\frac{n(n+1)}{12}\left[3n^2 + 23n + 34\right]$ 3. $\frac{1}{4} n(n+1)(n+2)(n+3)$

TERMINAL EXERCISE

1. (a) 1640 (b) 728

2. (a) $2^{n+1} - n - 2$ (b) $\frac{1}{12} - \frac{1}{4\,(2n+1)\,(2n+3)}$ (c) $1 - \frac{1}{(n+1)^2}$

3. $\frac{n}{3}(4n^2 - 1)$ 4. $\frac{1}{2}(3^n + 8n - 1)$

5. $\frac{5}{4} + \frac{15}{16}\left(1 - \frac{1}{5^{n-1}}\right) - \frac{3n-2}{4.(5^{n-1})}$ 6. $\frac{2n\,(n+1)\,(2n+1)}{3}$